OPERATOR SPACE STRUCTURES and the SPLIT PROPERTY II
The canonical non-commutative L^2 embedding

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ABSTRACT. A characterization of the split property for an inclusion $N \subset M$ of W^*-factors with separable predual is established in terms of the canonical non-commutative L^2 embedding considered in [4, 5]
$$\Phi_2 : a \in N \to \Delta_{M,\Omega}^{1/4} a\Omega \in L^2(M,\Omega)$$
associated with an arbitrary fixed standard vector Ω for M. This characterization follows an analogous characterization related to the canonical non-commutative L^1 embedding
$$\Phi_1 : a \in N \to (\cdot\Omega, J_{M,\Omega} a\Omega) \in L^1(M,\Omega)$$
also considered in [4, 5] and studied in [20]. The split property for a Quantum Field Theory is characterized by equivalent conditions relative to the non-commutative embeddings Φ_i, $i = 1, 2$, constructed by the modular Hamiltonian of a privileged faithful state such as e.g. the vacuum state. The above characterization would be also useful for theories on a curved space-time where there exists no a-priori privileged state.

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1. INTRODUCTION

The split property for inclusions of von Neumann algebras is of particular interest from a theoretical viewpoint ([4, 12, 13]) although it was introduced and intensively studied for the various applications in Quantum Field Theory, see [3, 6, 8, 39]. For the physical applications the relations of the split property with several nuclearity conditions is also of main interest, see [5, 7, 9].
The approach involving the study of properties of suitable sets (or equally well suitable maps) was firstly introduced by Haag and Swieca with the aim of characterizing those physical theories which are asymptotically complete. They conjectured that an asymptotically complete theory should satisfy a compactness criterion, see [22]. Motivated by

this approach, the split property was considered and a weak form of the celebrated Noether theorem was established also for a quantized theory, see [6]. Moreover, for a theory with the split property, the Haag-Swieca compactness criterion is automatically satisfied [5]. A nuclearity condition was proposed in [7], as a stronger condition than the split property. A very relevant consequence of nuclearity is that theories satisfying this property have a decent thermodynamical behavior, see [8].

To make the connection between the split property and the various nuclearity conditions more transparent, canonical non-commutative embeddings $M \hookrightarrow L^p(M)$, $p = 1, 2$ were considered firstly in [4]. Given an inclusion $N \subset M$ of W^*-algebras with M acting standardly on the Hilbert space $\mathcal{H}$ with a cyclic and separating vector Ω, one can consider the following embeddings, canonically constructed via the modular operators Δ, J associated to Ω, see [4].

$$\Phi_1 : a \in N \to (\cdot\Omega, Ja\Omega) \in M_*,$$

$$\Phi_2 : a \in N \to \Delta^{1/4} a\Omega \in \mathcal{H}.$$

In [4] it has been shown that, when $N \subset M$ is a factor-subfactor inclusion, the nuclearity condition for any of the above maps insures the split property for the inclusion $N \subset M$. Conversely, if $N \subset M$ is a split inclusion, these maps are nuclear for a dense set of cyclic separating vectors for M, see [4], Section 2.

Unfortunately the nuclearity condition is a stronger condition than the split property if some privileged state (as e.g. the vacuum) is kept fixed. Hence a complete characterization of the split property in terms of properties of the L^p embeddings would be desirable. This approach has been followed in [20] for the L^1 embedding where it has been shown that the metrically nuclear condition for Φ_1 characterize in a complete way the split property. In this case canonical operator space structures on the involved normed spaces play a crucial role, see [20], Section 2.

The aim of this work is to establish a complete characterization of the split property in terms of properties of the non-commutative embedding Φ_2. As an application one gets another characterization of the split property in Quantum Field Theory even if the local algebras of observables have a non-trivial center, see e.g. [3, 27]. Therefore, in all the interesting cases arising from Quantum Field Theory, the our characterization is available. To conclude this introduction, we recall that the complete characterization of the split property in terms of the above embeddings Φ_i, $i = 1, 2$ could be of interest also for theories living on curved space-time where there exists no a-priori privileged state as the vacuum ([41]). In this way the split property is directly

stated in terms of properties relative to the folium of states which are of interest for the theory.

This paper is organized as follows.

For the convenience of the reader we devote a preliminary part to resume some basic ideas on operator spaces considered by several authors, see [2, 14, 15, 16, 28, 35]. A particular attention will be reserved to the structures recently studied by Pisier such as the Hilbert OH structure ([31, 33]) and the non-commutative vector-valued L^P spaces ([32, 34]). We also describe the ideals $\mathcal{F}_p(E, F)$, $p \geq 1$, of the *p-factorable* maps between operator spaces E, F considered in [21]. Successively, following [4], we study some basic properties of the above non-commutative L^p embeddings $\Phi_i : M \to L^p(M)$, $p = 1, 2$ relative to a W^*-algebra M. As it has been well explained in [4] for a factor-subfactor inclusion $N \subset M$, the extendibility of the above embeddings, when restricted to the subfactor N, is related to the split property for the given inclusion. We connect the extendibility property of the above embeddings to a weaker property, the *quasi-split* property, in the case of general inclusions of W^*-algebras. The quasi-split property can also be stated using the language of Connes correspondences [10, 11].
A further section is devoted to study some extendibility properties of completely bounded normal maps of W^*-algebras with values into a type I factor as well as binormal bilinear forms constructed in a natural way considering pairings

$$a \otimes b \in M_1 \otimes M_2 \to \Psi_1 \times \Psi_2(a \otimes b) := (\Psi_1(a), \overline{\Psi_2(b)}) \in \mathbb{C}$$

where $\Psi_i : M_i \to \mathcal{H}$ are normal maps of W^*-algebras with values in a common Hilbert space $\mathcal{H}$. These results enable us to prove the announced characterization of the split property in terms of the canonical non-commutative L^2 embedding Φ_2:

Let $N \subset M$ be an inclusion of W^*-factors with separable predual and $\omega \in M_*$ a faithful state.
The inclusion $N \subset M$ is a split inclusion if and only if the canonical L^2 embedding

$$\Phi_{2,\omega\lceil N} : a \in N \to \Delta^{1/4} a \Omega \in L^2(M)$$

is 2-factorable as a map between the operator spaces N and $L^2(M)$ where N has the natural operator space structure as a C^*-algebra and $L^2(M)$ is equipped with the Pisier structure OH.

As an immediate corollary we have a geometrical characterization of the split property in terms of the "shape" of the image of the unit ball $\Phi_2(N_1) \subset L^2(M)$ under Φ_2. Another characterization of the split property suitable for the applications to Quantum Field Theory is so stated.

We conclude with a section containing comments and problems concerning the natural embeddings $\Phi_p : a \in A \rightarrow L^p(M)$ for a generic $p \geq 1$ canonically associated to the inclusion $N \subset M$.

In this paper we deal only with W^*- algebras with separable predual and all the operator spaces are considered to be complete if it is not otherwise specified. For the general theory of operator algebras we refer the reader to the celebrated texts [36, 38, 40].

2. STRUCTURES OF OPERATOR SPACES

For the reader's convenience we collect some ideas about the operator spaces which we need in the following. Details and proofs can be found later.

2.1. **Operator spaces.** We start with a normed space E, E_1 will be its (closed) unit ball. Let $\| \ \|_n$ be a sequence of norms on $\mathbb{M}_n(E)$, the space of $n \times n$ matrices with entries in E. For $a, b \in \mathbb{M}_n$ these norms satisfy

$$\|avb\|_n \leq \|a\| \ \|b\| \ \|v\|_n,$$
$$\|v_1 \oplus v_2\|_{n+m} = \max\{\|v_1\|_n, \|v_2\|_m\} \tag{1}$$

where the above products are the usual row-column ones. This space with the above norms is called an (abstract)*operator space*. If $T : E \in F$, $T_n : \mathbb{M}_n(E) \in \mathbb{M}_n(F)$ are defined as $T_n := T \otimes \text{id}$. T is said to be *completely bounded* if $\sup \|T_n\| = \|T\|_{cb} < +\infty$; $\mathcal{M}(E, F)$ denotes the set of all the completely bounded maps between E, F. It is an important fact (see [35]) that a linear space E with norms on each $\mathbb{M}_n(E)$ has a realization as a concrete operator space i.e. a subspace of a C^*-algebra, if and only if these norms satisfy the properties in (1). Given an operator space E and $f = \mathbb{M}_n(E^*)$, the norms

$$\|f\|_n = \sup\{\|(f(v))_{(i,k)(j,l)}\| : v \in \mathbb{M}_m(E)_1, \ m \in \mathbb{N}\}$$
$$(f(v))_{(i,k)(j,l)} = f_{ij}(v_{kl}) \in \mathbb{M}_{mn} \tag{2}$$

determines an operator structure on E^* that becomes itself an operator space ([2]). The linear space $\mathbb{M}_I(E)$, I any index set, is also of interest here. $\mathbb{M}_I(E)$ is in a natural manner an operator space via inclusion $\mathbb{M}_I(E) \subset \mathcal{B}(\mathcal{H} \otimes \ell^2(I))$ if E is realized as a subspace of $\mathcal{B}(\mathcal{H})$. Of

interest is also the definition of $\mathbb{K}_I(E)$ as those elements $v \in \mathbb{M}_I(E)$ such that $v = \lim_{\Delta} v^{\Delta}$, where Δ is any finite truncation. Obviously $\mathbb{M}_I(\mathbb{C}) \equiv \mathbb{M}_I = \mathcal{B}(\ell^2(I))$ and $\mathbb{K}_I(\mathbb{C}) \equiv \mathbb{K}_I = \mathcal{K}(\ell^2(I))$, the set of all the compact operators on $\ell^2(I)$. For E complete we remark the bimodule property of $\mathbb{M}_I(E)$ over $\mathbb{K}_I$ because, for $\alpha \in \mathbb{K}_I$, $\alpha^{\Delta} v$, $v\alpha^{\Delta}$ are Cauchy nets in $\mathbb{M}_I(E)$ and its limits define unique elements αv, $v\alpha$ that can be calculated via the usual row-column product.

Given an index set I, we can define a map $\mathcal{X} : \mathbb{M}(V^*) \to \mathcal{M}(V, \mathbb{M}_I)$ which is a complete isometry, given by

$$(\mathcal{X}(f)(v))_{ij} := f_{ij}(v). \tag{3}$$

Moreover, if $f \in \mathbb{K}(V^*)$, $\mathcal{X}(f)$ is norm limit of finite rank maps, so $\mathcal{X}(\mathbb{K}_I(V^*)) \subset \mathcal{K}(V, \mathbb{K}_I)$, see [17], Section 3.

Remarkable operator space structures on a Hilbert space H was introduced and studied in [18] via the following identification

$$\mathbb{M}_{p,q}(H_c) := \mathcal{B}(\mathbb{C}^q, H^p),$$
$$\mathbb{M}_{p,q}(H_r) := \mathcal{B}(\overline{H}^q, \mathbb{C}^p)$$

which define on H the column and row structures H_c, H_r respectively. Recently the theory of complex interpolation has been developped by Pisier for operator space as well. Successively a highly symmetric structure on a Hilbert space H has been considered in [31, 33], the OH structure. It can be viewed as an interpolating structure between H_c, H_r:

$$OH(I) := (H_c, H_r)_{1/2}$$

where the cardinality of the index set I is equal to the (Hilbert) dimension of H. The explict descricption of the OH structure, contained in [33], Theorem 1.1, is useful here and we report it in the next section for the sake of completness together with some others elementary properties.

2.2. **Tensor products between operator spaces.** Let E, F be operator spaces, tensor norms on the algebraic tensor product $E \otimes F$ can be considered. They make (the completions of) $E \otimes F$ operator spaces themselves. The (operator) projective and spatial tensor products $E \otimes_{max} F$, $E \otimes_{min} F$ ([16, 17]) are of interest here together with the Haagerup tensor product $E \otimes_h F$ ([18]). As F is complete, we have the completely isometric inclusion $E^* \otimes_{min} F \subset \mathcal{M}(E, F)$. Also of interest is the following complete identification

$$\mathbb{K}(E) = E \otimes_{min} \mathbb{K}_I \tag{4}$$

for each operator space E. The following important result concern the description of the predual of a W^*-tensor product in terms of the preduals of its individual factors.

Theorem 1. *let N, M be any W^*-algebras. The predual $(A\overline{\otimes}B)_*$ is completely isomorphic to the projective tensor product $A_* \otimes_{max} B_*$.*

The detailed proof of the above result can be found in [17], Section 3. The above important result has a central role in the complete characterization of the split property in terms of the L^1 embedding Φ_1, see [20].

2.3. **The metrically nuclear maps.** In [19] and, independently in [20], the class of the metrically nuclear maps $\mathcal{D}(E, F)$ between operator spaces E, F, has been introduced and studied. They can be defined as

$$\mathcal{D}(E, F) := E^* \otimes_{max} F/\mathrm{Ker}\mathcal{X}$$

where $\mathcal{X}$ is the map (3) which is a complete quotient map when restricted to $E^* \otimes_{max} F$. Also a geometrical characterization can be provided. Namely, an injective operator $T \in \mathcal{D}(E, F)$ is completely characterized by the shape of the set $T(E_1) \subset F$, see [20], Section 2. The spaces $\mathcal{D}(E, F)$ constituite, in the language of [21, 30], an *operator ideal* which have themselves a canonical operator space structure, see [19, 20].

2.4. **The non-commutative vector-valued L^p spaces.** The vector-valued non-commutative L^p spaces has been introduced and intensively studied by Pisier [32, 34] by considering, for an operator space E, the compatible couple of operator spaces

$$(S_1(H) \otimes_{max} E, S_\infty(H) \otimes_{min} E)\,; \tag{5}$$

$S_1(H)$, $S_\infty(H)$ are the trace class and the class of all compact operators acting on the Hilbert space H. Then the vector-valued non-commutative L^p spaces $S_p[H, E]$ can be defined as the interpolating spaces relative to the compatible couple (5)

$$S_p[H, E] := (S_\infty(H) \otimes_{min} E, S_1(H) \otimes_{max} E)_\theta \tag{6}$$

where $\theta = 1/p$. As it has been described in [34] Theorem 1.1, the non-commutative vector-valued L^p spaces can be viewed as Haagerup tensor products between the interpolating space structures on a Hilbert spaces

$$S_p[H, E] = R(1 - \theta) \otimes_h E \otimes_h \overline{R(\theta)} \tag{7}$$

where $R(\theta) := (H_r, H_c)_\theta$ with $\theta = 1/p$. In particular we have for $p = 2$

$$S_2[H, E] = OH \otimes_h E \otimes_h \overline{OH}. \tag{8}$$

The concrete structure of all the $S_p[H, E]$ can be derived from [32], Theorem 2. When there is no matter of confusion (i.e. if the Hilbert space is kept fixed), we simply write $S_p[E]$ instead of $S_p[H, E]$.

For the reader's convenience we conclude with a result quite similar to that contained in [17] Proposition 3.1 which will be useful in the following, its proof can be found in [21]. We start with a Hilbert space H of (Hilbert) dimension given by the cardinality of the index set I and make the identification $H \equiv \ell^2(I)$.

Proposition 1. *An element u in $S_p[H, E] \subset \mathbb{M}_I(E)$ satisfies $\|u\|_{S_p[H,E]} < 1$ iff there exists elements $a, b \in S_{2p}(H) \subset \mathbb{M}_I(E)$ with and $v \in \mathbb{M}_I(E)$ with $\|a\|_{S_{2p}(H)} = \|b\|_{S_{2p}(H)} = 1$ and $\|v\|_{\mathbb{M}_I(E)} < 1$ such that*

$$u = avb.$$

Furthermore one can choose $v \in \mathbb{K}_I(E)$.

3. THE PISIER OH HILBERT SPACE

Here we collect some results concerning the Pisier OH Hilbert space which we need in the following.

We start with the concrete description of the OH structure whose proof can be easily recovered from [31], Theorem 1.1.

Proposition 2. *Let $OH \equiv OH(I)$ be a Hilbert space equipped with the OH structure for a fixed index set I and $x \in \mathbb{M}_n(OH)$. Then*

$$\|x\|_{\mathbb{M}_n(OH)} = \|(x_{ij}, x_{kl})\|_{\mathbb{M}_{n^2}}^{1/2}$$

where the entries of the above numerical matrix are as those in (2).

Proof. If $OH \subset \mathcal{B}(\mathcal{H})$ then

$$\|x\|_{\mathbb{M}_n(OH)} = \|x\|_{\mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)}.$$

We consider x in the dense set of $\mathbb{M}_n(OH)$ consisting of finite linear combinations of elements of the form $x = a \otimes e_k$ where $a \in \mathbb{M}_n$, $\{e_k\}_{k \in I}$ an ortonormal basis for OH. We have

$$x_{ij} = \sum_{k \in I} (x_{ij}, e_k) e_k$$

where the coefficients in the above sum are all zero except a finitely many of them. Then, by [33], Theorem 1.1 part (iii), we obtain

$$\|x\|_{\mathbb{M}_n(OH)} = \|\sum_{k \in I} a_k \otimes e_k\|_{\mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)} = \|\sum_{k \in I} a_k \otimes \overline{a}_k\|_{\mathcal{B}(\mathbb{C}^n \otimes \overline{\mathbb{C}}^n)}^{1/2}$$

where the numerical matrices a_k are defined as $(a_k)_{ij} = (x_{ij}, e_k)$. Therefore, as $(a_k \otimes \bar{a}_k)_{(i,l)(j,m)} = (x_{ij}, e_k)(e_k, x_{lm})$ we get

$$\|x\|_{\mathbb{M}_n(OH)} = \|\sum_{k \in I}(x_{ij}, e_k)(e_k, x_{lm})\|^{1/2} \equiv \|(x_{ij}, x_{lm})\|^{1/2}_{\mathbb{M}_{n^2}}.$$

The proof now follows by a standard continuity argument.   □

The following Proposition is a non-commutative version of the Cauchy-Schwarz inequality and follows from a result due to Haagerup.

Proposition 3. *Let $OH \equiv OH(I)$ be as in the last Proposition and $x \in \mathbb{M}_m(OH)$, $y \in \mathbb{M}_n(OH)$. Then*

$$\|(x, y)\|_{\mathbb{M}_{mn}} \leq \|x\|_{\mathbb{M}_m(OH)}\|y\|_{\mathbb{M}_n(OH)}.$$

Proof. We proceed as in the above Proposition.

$$\|(x, y)\|_{\mathbb{M}_{mn}} = \|\sum_\sigma (x, e_\sigma)(e_\sigma, y)\| = \|\sum_\sigma a_\sigma \otimes \bar{b}_\sigma\|_{\mathcal{B}(\mathbb{C}^m \otimes \bar{\mathbb{C}}^n)}.$$

Now we apply [23], Lemma 2.4 and obtain

$$\|(x, y)\|_{\mathbb{M}_{mn}} \equiv \|\sum_\sigma a_\sigma \otimes \bar{b}_\sigma\|_{\mathcal{B}(\mathbb{C}^m \otimes \bar{\mathbb{C}}^n)}$$

$$\leq \|\sum_\sigma a_\sigma \otimes \bar{a}_\sigma\|^{1/2}_{\mathcal{B}(\mathbb{C}^m \otimes \bar{\mathbb{C}}^m)}\|\sum_\sigma b_\sigma \otimes \bar{b}_\sigma\|^{1/2}_{\mathcal{B}(\mathbb{C}^n \otimes \bar{\mathbb{C}}^n)}$$

$$\equiv \|x\|_{\mathbb{M}_m(OH)}\|y\|_{\mathbb{M}_n(OH)}$$

which is the proof.   □

4. The class of factorable maps

In this Section we resume the main properties of a class $\mathcal{F}_p(E, F) \subset \mathcal{K}(E, F)$, $1 \leq p < +\infty$, of linear maps between operator spaces E, F which are limits of finite rank maps. These maps are obtained considering operators arising in a natural way from the Pisier non-commutative vector-valued L^p spaces $S_p[H, E]$ and has been called the *p-factorable* maps in [21]. We also report for $p = 2$, a geometrical description of the image $T(E_1) \subset F$ of the unit ball of E under an injective 2-factorable map. Details and proof can be found in [21].

4.1. The *p*-factorable maps.

Definition 1. Let E, F be operator spaces and $1 \leq p < +\infty$. A linear map $T : E \in F$ will be called *p-factorable* if there exists elements $b \in S_p[E^*]$, $A \in \mathcal{M}(S_p, F)$ such that T factorizes as

$$T = AB$$

where $B = \mathcal{X}(b) \in \mathcal{M}(E, S_p)$ and $\mathcal{X}$ is the map (3).
We also define

$$\varphi_p(T) := \inf\{\|A\|_{cb}\|b\|_{S_p[E^*]}\}$$

where the infimum is taken on all the factorization for T as above.
The class of all the p-factorable maps between E, F will be denoted as
$\mathcal{F}_p(E, F)$.

In the above definition we have supposed $H \cong \ell^2$ without loss of
generality, see [21], Remark 1.

Remark 1. We always have $\mathcal{F}_p(E, F) \subset \mathcal{K}(E, F)$ as $\mathcal{X}(\mathbb{K}_I(E^*)) \subset \mathcal{K}(E, \mathbb{K}_I)$.

In [21] we have shown that $(\mathcal{F}_p(E, F), \varphi_p)$, $1 \leq p < +\infty$ are quasi-
normed complete vector space. Moreover we also have the ideal prop-
erty for the above factorable maps, that is $RST \in \mathcal{F}_p(E_0, F_0)$ and

$$\varphi_p(RST) \leq \|R\|_{cb}\varphi_p(S)\|T\|_{cb}$$

whenever E_0, E, F, F_0 are operator spaces and $T : E_0 \to E$, $S : E \to F$, $R : F \to F_0$ linear maps with $T \in \mathcal{M}(E_0, E)$, $S \in \mathcal{F}_p(E, F)$, $R \in \mathcal{M}(F, F_0)$ respectively.
Actually the case with $p = 2$ is particular: $(\mathcal{F}_2(E, F), \varphi_2)$ is a Banach
space for every operator space E, F.
Summarizing we have

Theorem 2. *([21] Theorem 2)* $(\mathcal{F}_p, \varphi_p)$, $1 \leq p < p$ *are quasi-normed
operator ideals whereas* $(\mathcal{F}_2, \varphi_2)$ *is a Banach operator ideal.*

4.2. **A geometrical description.** Analogously to the metrically nu-
clear operator setting [20], we give a suitable geometrical description
for the range of a 2-factorable injective map. An analogous description
can be stated also for a p-factorable injective map, see [21].
We start with an absolutely convex set Q in an operator space E and
indicate with V its algebraic span. Consider a sequence $\mathcal{Q} \equiv \{Q_n\}$ of
sets such that

(i) $Q_1 \equiv Q$ and each Q_n is an absolutely convex absorbing set of
$\mathbb{M}_n(V)$ with $Q_n \subset \mathbb{M}_n(Q)$;
(ii) $Q_{m+n} \cap (\mathbb{M}_m(V) \oplus \mathbb{M}_n(V)) = Q_m \oplus Q_n$;
(iii) for $x \in Q_n$ then $x \in \lambda Q_n$ implies $bx \in \lambda Q_n, xb \in \lambda Q_n$ where
$b \in (\mathbb{M}_n)_1$.

We say that a (possibly) infinite matrix f with entries in the algebraic
dual of V has finite $\mathcal{Q}$-norm if

$$\|f\|_{\mathcal{Q}} \equiv \sup\{\|f^\Delta(q)\| : q \in Q_n; \ n \in \mathbb{N}; \ \Delta\} < +\infty$$

where f^Δ indicates an arbitrary finite truncation corresponding to the finite set Δ; the numerical matrix $f^\Delta(q)$ has entries as those in (1.2).

Definition 2. An absolutely convex set $Q \subset E$ is said to be $(2, \mathcal{Q})$-*factorable* (where $\mathcal{Q}$ is a fixed sequence as above) if there exists matrices $\alpha, \beta \in S_4$ and a (possible infinite) matrix f of linear functionals as above with $\|f\|_\mathcal{Q} < +\infty$ such that, for each $x \in \mathcal{Q}_n$, one has

$$\|x\|_{\mathbb{M}_n(E)} \le C\|\alpha f(x)\beta\|_{\mathbb{M}_n(S_2)}. \tag{9}$$

In the above case corresponding to $p = 2$ we call a $(2, \mathcal{Q})$-factorable set simply $\mathcal{Q}$-factorable and omit the dependence on the sequence $\mathcal{Q}$ if there is no matter of confusion.

One can easily see that a $\mathcal{Q}$-factorable set is relatively compact, hence bounded in the norm topology of E and therefore Q, together the Minkowski norms determinated by the Q_n's on $\mathbb{M}_n(V)$, is a (not necessarily complete) operator space. Moreover the canonical immersion $V \overset{i}{\hookrightarrow} E$ is completely bounded when V is equipped with the operator structure determined by the sequence $\mathcal{Q}$.

We now consider an injective completely bounded operator $T : E \to F$ and the sequence $\mathcal{Q}_T$ given by $\mathcal{Q}_T = \{T_n(\mathbb{M}_n(V)_1)\}$. For such sequences the properties (i)–(iii) are automatically satisfied and, if $T(V_1)$ is $\mathcal{Q}_T$-factorable, we call it simply T-factorable and indicate the $\mathcal{Q}_T$-norm of a matrix of functional f by $\|f\|_T$. According to some interesting well-known cases, we have a description in terms of the geometrical property of the range of certain 2-factorable operators.

Theorem 3. *Let E, F operator spaces and $T : E \to F$ a completely bounded injective map. $T \in \mathcal{F}_2(E, F)$ iff $T(V_1)$ is a T-factorable set in F.*

As in the case relative to the metrically nuclear maps, the definition of a factorable set may appear rather involved; this is due to the fact that the inclusion $\mathbb{M}_n(V)_1 \subset \mathbb{M}_n(V_1)$ is strict in general but, for an injective completely bounded operator T as above, the T-factorable set $T(V_1)$ is intrinsecally defined in terms of T.

5. Inclusions of W^*-algebras and some related canonical embeddings

We consider an inclusion $N \subset M$ of W^*-algebras with separable preduals where M acts standardly on $\mathcal{H}$ with cyclic separating vector Ω. Let Δ, J be the Tomita's modular operator and conjugation relative

to Ω respectively. Together with M we can consider its opposite algebra M° which is $*$-isomorphic to M' via

$$x \in M^\circ \in j(x) \equiv Jx^*J \in M';$$

the inverse is given in the same way where M° is identified with M as a linear space. The appearence of the opposite algebra M° will be clear later.

Following [4], is of interest to consider the L^1 non-commutative embedding

$$\Phi_1 : b \in M \in (\cdot\Omega, Jb\Omega) \in L^1(M^\circ) \tag{10}$$

where we identify in a natural manner $L^1(M^\circ)$ with $(M^\circ)_*$. The following non-commutative L^2 embedding is also of particular interest ([4, 5]).

$$\Phi_2 : b \in M \in \Delta^{1/4}b\Omega \in L^2(M^\circ) \tag{11}$$

where $L^2(M^\circ) \equiv \mathcal{H}$.

If is not otherwise specified, all the Hilbert spaces appearing in the sequel should be considered as endowed with the Pisier OH structure as operator spaces.

It is a simple but important fact that the embeddings considered above are completely positive in a sense which we are going to explain.

Let A, B be C^*-algebras. A linear map $\Psi : A \to B^*$ is said to be *completely positive* if all the maps

$$\Psi \otimes \mathrm{id} : \mathbb{M}_n(A) \to \mathbb{M}_n(B^*)$$

are positive that is $(\Psi \otimes \mathrm{id})(a)$ is a positive element of $\mathbb{M}_n(B)^*$ whenever a is a positive element of $\mathbb{M}_n(A)$. It is easy to show that the complete positivity for Ψ is equivalent to the request that the linear form ω_Ψ given by

$$\omega_\Psi(a \otimes b) := \Psi(a)(b)$$

define a positive form on the algebraic tensor product $A \otimes B$.

For the L^2 case we should start with a W^*-algebra and consider a hierarchy of self-polar cones as follows. Let M be a W^*-algebra with a normal semifinite faithful weight φ. If M is represented standardly on $L^2(M, \varphi) \equiv L^2(M)$ we can consider a hierarchy of self-polar cones $\mathcal{P}_n \subset \mathbb{M}_n(L^2(M))$, $n \in \mathbb{N}$, naturally associated to the normal semifinite faithful weight $\varphi \otimes \mathrm{Tr}_n$ on the W^*-algebras $\mathbb{M}_n(M)$, see e.g. [38] 10.23.

A characterization of when a hierarchy of self-polar cones $\mathcal{C}_n \subset \mathbb{M}_n(\mathcal{H})$, $n \in \mathbb{N}$ is associated to a W^*-algebra as above is given in [37].

Let N be another W^*-algebra and $\Phi : N \to L^2(M)$ a linear map. Φ will be said *completely positive* if the map $\Phi_n : \mathbb{M}_n(N) \to \mathbb{M}_n(L^2(M))$ is positive that is $\Phi_n(\mathbb{M}_n(N)_+) \subset \mathcal{P}_n$ for each $n \in \mathbb{N}$.

It is also of main interest to consider the normality property of linear maps $T : M \to X$ of a W^*-algebra M with values into an arbitrary linear space X.

Let M be a W^*-algebra, X a linear space with algebraic dual X' and $F \subset X'$ a separating set of functionals. A continuous linear map $T : M \in X$ is said to be *normal (singular)* w.r.t. F according to the functionals $f \circ T$ on M are normal (singular) for every $f \in F$ (for the definition of normal and singular functionals on a von Neumann algebra see [40]). Precisely, the F-normal maps are just those which are $(\sigma(M, M_*), \sigma(X, F))$–continuous, whereas the singular ones are those such that $T'(F) \subset M_*^\perp$ where T' is the transpose map of T. From this definition we get at once that a map which is normal and singular at the same time must be the zero map, see [40]. If X is also a W^*-algebra, it is usual to take $F \equiv M_*$; if X is a normed space (which is not a W^*-algebra) and $F = X^*$, the topological dual, we call a normal map w.r.t. X^* simply normal as well.

The normal property of the above non-commutative embeddings are summarized in the following proposition (see [4]).

Proposition 4. *The non-commutative embeddings (10), (11) are completely positive and normal.*

Proof. It is easy to show that Φ_i, $i = 1, 2$ are completely positive and that Φ_1 is normal so it remains to verify that Φ_2 is normal too. Suppose that $\eta \in L^2(M^\circ)$, we obtain

$$|(\Delta^{1/4}x\Omega, \eta)| \leq \|\eta\| \|\Delta^{1/4}x\Omega\| = \|\eta\| (\Delta^{1/2}x\Omega, x)^{1/2}$$

$$\leq \|\eta\| (\|x\Omega\| \|x^*\Omega\|)^{1/2} \leq \frac{1}{\sqrt{2}} \|\eta\| (\|x\Omega\|^2 + \|x^*\Omega\|^2)^{1/2}$$

which shows that Φ_2 is continuous in the strong* operator topology hence $\sigma(M, M_*)$–continuous, see [40], Theorem II.2.6. $\square$

A completely positive map between C^*-algebras is automatically completely bounded, see e.g. [28]; concerning the general case (i.e. when the image space is not a C^*-algebra or merely an operator system)

it seems to be no relation between complete positivity and complete boundedness. However the above embeddings are also completely bounded.

Proposition 5. *The non-commutative embeddings (10), (11) are completely bounded.*

Proof. The first case is just [20], Proposition 3.1, so we prove the remaining one. By Proposition 2, we obtain for an element $a \in \mathbb{M}_n(M)$

$$(\Delta^{1/4} a_{ij}\Omega, \Delta^{1/4} a_{kl}\Omega) = (a_{ij}\Omega, j(a_{kl}^*)\Omega) = (a_{ij} j(a_{kl})\Omega, \Omega) \equiv \omega_{n^2}(b)$$

where $b \in \mathbb{M}_{n^2}(\mathcal{B}(L^2(M)))$ has norm equal to $\|a\|^2$. Taking the supremum on the unit ball of $\mathbb{M}_n(M)$ we obtain the assertion. $\square$

6. The quasi-split property for inclusions of W^*-algebras

This Section follows Section 1 of [4] where the split property for inclusions of factors is connected with the extendibility of the canonical non-commutative embeddings Φ_i, $i = 1, 2$ whose early properties have been summarized in the last Section. Here we treat the general case of inclusions of W^*-algebra at all. The results contained in this Section could be naturally applied to inclusions arising from Quantum Field Theory where, under general assumptions, the algebras of the local observables are typically type III algebras with (a-priori) a non-trivial center [3, 27]. This framework covers also the commutative case which has been treated in some detail in [4], Section 3. We note that the subjects contained in this Section have a natural description in terms of Connes *correspondence* so we adopt somewhere the correspondence language in the following.

We start with an inclusion $N \subset M$ of W^*-algebras always with separable predual, and suppose that M is represented in standard form on $L^2(M)$. We take a normal faithful state $\omega \in M_*$ and consider the Tomita operators J, Δ relative to it.

Definition 3. The inclusion $N \subset M$ is said to be *quasi-split* if the map

$$a \otimes b \in N \otimes M^\circ \to aJb^*J \in \mathcal{B}(\mathcal{H}) \qquad (12)$$

extends to a normal homomorphism η of $N \overline{\otimes} M^\circ$ onto all of $N \vee M'$.

We remark that, since the standard representation is unique up unitary equivalence, the quasi-split property is really an intrinsic property of the inclusion.

Let N, M be W^*-algebras, a $N - M$ *correspondence* is a separable Hilbert space $\mathcal{H}$ which is a $N - M$-bimodule such that the (commuting)

left and right actions of N, M are normal.

For the basic facts about the correspondence we remand the reader to [10, 11].

The standard representation of M gives rise to a correspondence which is unique up unitary equivalence, just the identity $M - M$ correspondence $_M\mathrm{id}_M$ on $L^2(M)$ determined by the map (12). If we have an inclusion $N \subset M$ of W^*-algebras we obtain a (uniquely determined) $N - M$ correspondence if one restrict $_M\mathrm{id}_M$ to N on the left. We indicate this one as $_N\mathrm{id}_M$

Definition 4. A $N - M$ correspondence σ is said to be *split* if there exists normal faithful representations π, π° of N, M° on Hilbert spaces $\mathcal{H}_\pi$, $\mathcal{H}_{\pi^\circ}$ respectively such that σ is unitarily equivalent to $\pi \otimes \pi^\circ$ ($\sigma \cong \pi \otimes \pi^\circ$ for short).

The term *coarse* is used in [10] for the case when π, π° are the standard representations of N, M° respectively.

Now we show that the quasi-split property can be viewed as a property of $_N\mathrm{id}_M$.

Proposition 6. *Let $N \subset M$ an inclusion of W^*-algebras.*
The following assertion are equivalent.
(i) $N \subset M$ is a quasi-split inclusion.
(ii) $_N id_M \prec \sigma$
where σ is a $N - M$ split correspondence and $\prec$ means the containment of representations.

Proof. $(i) \Rightarrow (ii)$ If the map (12) extends to a normal homomorphism η of $N \overline{\otimes} M^\circ$ onto $N \vee M'$ then $\eta = \eta_1 \circ \eta_2 \circ \eta_3$ where η_3 is an amplification, η_2 an induction and η_1 a spatial isomorphism. Hence
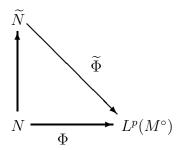
$$\eta(a \otimes b) = v^*(a \otimes b \otimes I)v$$

where $v : L^2(M) \to L^2(M) \otimes L^2(M) \otimes \mathcal{H}$ is an isometry that is the $N - M$ correspondence $_N\mathrm{id}_M$ is contained in a split one.

$(ii) \Rightarrow (i)$ If $_N\mathrm{id}_M$ is a subcorrespondence of a split one we have then a subrepresentation of the normal representation $\pi \otimes \pi^\circ$ of $N \overline{\otimes} M^\circ$, see [40], Theorem IV.5.2. Hence $_N\mathrm{id}_M$ uniquely defines a normal representation of $N \overline{\otimes} M^\circ$ that is a normal map η of $N \overline{\otimes} M^\circ$ onto $N \vee M'$ which extends (12). $\qquad\square$

When a faithful state $\omega \in M_*$ is kept fixed, one can construct the non-commutative embeddings Φ_1, Φ_2 given in (10), (11) and, following [4], one can study the extendibility properties of Φ_i, $i = 1, 2$.

A completely positive normal map $\Phi : N \to L^p(M^\circ)$, $p = 1, 2$, is said to be *extendible* (according with [4], pag 236) if, whenever $\widetilde{N} \supset N$ is another W^*-algebra with separable predual, there exists a completely positive normal map $\widetilde{\Phi}$ which yields commutative the following diagram:
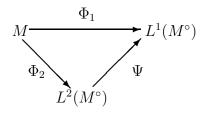
$$
\begin{array}{ccc}
\widetilde{N} & & \\
\uparrow & \searrow^{\widetilde{\Phi}} & \\
N & \xrightarrow{\Phi} & L^p(M^\circ)
\end{array}
$$

The following Theorem is the natural extension of [4], Proposition 1.1.

Theorem 4. *Let $N \subset M$ be an inclusion of W^*-algebras and $\omega \in M_*$ a fixed faithful state. Consider the embeddings Φ_i, $i = 1, 2$ constructed by ω as in (10),(11).*
The following statements are equivalent.
(i) $N \subset M$ is a quasi-split inclusion.
(ii) $\Phi_{1 \upharpoonright N} : N \to L^1(M^\circ)$ is extendible.
(iii) $\Phi_{2 \upharpoonright N} : N \to L^2(M^\circ)$ is extendible.

Proof. $(iii) \Rightarrow (ii)$ We can consider the following commutative diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\Phi_1} & L^1(M^\circ) \\
 _{\Phi_2}\searrow & & \nearrow_{\Psi} \\
 & L^2(M^\circ) &
\end{array}
$$

where Ψ given by

$$\Psi(x) := (\Delta^{1/4} \cdot \Omega, Jx)$$

is computed by the transpose map of Φ_2. So, if $\Phi_{2 \upharpoonright N}$ is extendible, it is easy to verify that $\Phi_{1 \upharpoonright N}$ is extendible too.

$(ii) \Rightarrow (i)$ Let F be a type I factor with separable predual containing N and $\widetilde{\Phi}_1$ the corresponding completely positive normal extension of Φ_1. Then

$$\widetilde{\varphi}(f \otimes b) := \widetilde{\Phi}_1(f)(b).$$

gives rise to a $F-M$ correspondence $(\sigma, \mathcal{H}_\sigma)$. Moreover the $F-M$ correspondence $(\sigma, \mathcal{H}_\sigma)$, when restricted to a N-M correspondence, contains $_N\text{id}_M$. The assertion now follows as the standard representation of M is faithful.

$(i) \Rightarrow (iii)$ It is enough to show that Φ_2 extends to a type I factor with separable predual containing N ([4], Proposition 1.1). Let $(\pi \otimes \pi^\circ, \mathcal{H}_\pi \otimes \mathcal{H}_{\pi^\circ})$ be a split correspondence containing $_N\text{id}_M$. Then there exists an isometry $v : L^2(M) \to \mathcal{H}_\pi \otimes \mathcal{H}_{\pi^\circ}$ with range projection $vv^* \in \pi(N)'\overline{\otimes}\pi^\circ(M^\circ)'$, such that, for $a \in N$, $b \in M^\circ$, we get

$$aJb^*J = v^*\pi(a) \otimes \pi^\circ(b)v.$$

As $vv^* \in \mathcal{B}(\mathcal{H}_\pi)\overline{\otimes}\pi^\circ(M^\circ)'$, it is easy to show that $v^*(\mathcal{B}(\mathcal{H}_\pi)\otimes I)v \in M$. Now we define $\widetilde{\Phi}_2 : \mathcal{B}(\mathcal{H}_\pi) \to L^2(M^\circ)$ as

$$\widetilde{\Phi}_2(f) := \Delta^{1/4}v^*(f \otimes I)v\Omega$$

which is a completely positive normal extension of Φ_2 to the type I factor $\mathcal{B}(\mathcal{H}_\pi)$.   □

We note that the Proposition 1.1 of [4] can be immediately recovered as a corollary of the above result.

Corollary 1. *If $N \subset M$ is an inclusion of W^*-factors, then the condition (i) in the last Theorem can be replaced by the a-priori stronger condition*
(i') $N \subset M$ *is a split inclusion.*

Proof. Let η be the normal map which extends (12) to all of $N\overline{\otimes}M$. If N, M, are both factors then $\text{Ker}\eta = \{0\}$ that is η is a normal isomorphism onto $N \vee M'$ but, in this case ([12], Corollary 1) that condition turn out to be equivalent to the split property.   □

As it has been explained in [4] for factor-subfactor inclusions, the above theorem gets a characterization of the quasi-split property in terms of the extendibility of the canonical embeddings Φ_1, Φ_2. Moreover the extendibility condition allows us to characterize the split property for inclusions of factors (or also of algebras in some interesting cases such those arising from Quantum Field Theory) directly in terms of properties of Φ_1, Φ_2 somewhat similar to the nuclear condition. This has been made in [20] for the map Φ_1. We continue this program also

for the L^2 embedding Φ_2 which is also of interest for the applications, see [5].

7. On some completely bounded maps of C^*-algebras and W^*-algebras

In this Section we analyze some extendibility properties of certain completely bounded maps or bilinear (binormal) form on C^* and W^*-algebras as well. These results will be crucial in the following. The results contained in this section are obtained under no separability conditions on the W^*-algebras which we are interested in.
We start with a result regarding the canonical structure of normal maps of a W^*-algebras with values in an arbitrary type I factor. This result directly follows from the analogous one contained in [1] and relative to C^*-algbras.

Proposition 7. *Let M be a W^*-algebra and $\mathcal{H}$ a Hilbert space. Suppose that a completely bounded normal map $\pi : M \to \mathcal{B}(\mathcal{H})$ is given.*
(i) There exists another Hilbert space $\mathcal{K}$, bounded operators $V_i : \mathcal{H} \to \mathcal{K}$, $i = 1, 2$ and a normal homomorphism $\rho : M \to \mathcal{B}(\mathcal{K})$ such that

$$\pi = V_1^* \rho(\cdot) V_2.$$

(ii) If $F \supset M$ is a type I factor containing M, then π extends to a completely bounded normal map $\widetilde{\pi} : F \to \mathcal{B}(\mathcal{H})$.

Proof. (i) π is completely bounded so there exists another Hilbert space $\mathcal{K}$, bounded maps $V, W : \mathcal{H} \to \mathcal{K}$ and an homomorphism $\sigma : M \to \mathcal{B}(\mathcal{K})$ such that

$$\pi = V^* \sigma(\cdot) W,$$

see [28],Theorem 7.4, or [1] for the original exposition. Moreover we can decompose σ in its normal and singular part $\sigma = \sigma_n \oplus \sigma_s$ so we can write for V, W

$$V = \begin{pmatrix} V_n \\ V_s \end{pmatrix}, \quad W = \begin{pmatrix} W_n \\ W_s \end{pmatrix}.$$

We get for π

$$\pi = W_n^* \sigma_n(\cdot) V_n + W_s^* \sigma_s(\cdot) V_s.$$

As π is normal, we have that $\pi - W_n^* \sigma_n(\cdot) V_n \equiv W_s^* \sigma_s(\cdot) V_s$ is, at the same time, normal and singular and must be the zero map so the first part is proved if one chooses $\rho := \sigma_n$.
(ii) As $N \subset F \equiv \mathcal{B}(\mathcal{L})$, the homomorphism ρ given in (i) can be written as $\rho = \rho_1 \circ \rho_2 \circ \rho_3$ where ρ_3 is an ampliation, ρ_2 an induction

and ρ_1 a spatial isomorphism. Namely there exists a suitable Hilbert space $\widehat{\mathcal{L}}$ and an isometry $U : \mathcal{K} \to \mathcal{L} \otimes \widehat{\mathcal{L}}$ such that

$$\rho(a) = U^*(a \otimes I)U.$$

Now, if $f \in \mathcal{B}(\mathcal{L})$, we define

$$\widetilde{\pi}(f) := V_1^* U^*(f \otimes I)UV_2$$

which is a completely bounded normal extension of π to all of F. $\square$

Another interesting situation is when one consider a pairing constucted in a natural manner by linear maps of C^*-algebras or W^*-algebras with values in a Hilbert space equipped with the Pisier OH structure as an operator space.

Let M_i, $i = 1, 2$ be C^*-algebras and Ψ_i, $i = 1, 2$ linear maps with range in a fixed Hilbert space $\mathcal{H}$. Then a bilinear form $\Psi_1 \times \Psi_2 : M_1 \otimes M_2 \to \mathbb{C}$ is defined as follows

$$(\Psi_1 \times \Psi_2)(a_1 \otimes a_2) := (\Psi_1(a_1), \overline{\Psi_2(a_2)}) \tag{13}$$

where we have made the usual (antilinear) identification between $\mathcal{H}$ and $\overline{\mathcal{H}}$. Suppose that $\mathcal{H}$ is endowed with the Pisier OH structure as an operator space. We have the following

Proposition 8. *If Ψ_i, $i = 1, 2$ are completely bounded then the linear form (13) extends to a bounded functional on the projective tensor product $M_1 \otimes_{max} M_2$.*
Moreover if M_i, $i = 1, 2$ are W^*-algebras and Ψ_i, $i = 1, 2$ are normal, then the linear form (13) extends to a binormal bounded form on all of $M_1 \otimes_{max} M_2$.

Proof. By [16] Theorem 3.2, the above bilinear form extends to a bounded functional on $M_1 \otimes_{max} M_2$ iff $\|\Psi_1 \times \Psi_2\|_{cb} < +\infty$. We get by Proposition 3

$$\|(\Psi_1(a), \overline{\Psi_2(b)}\|_{\mathbb{M}_{mn}} \leq \|\Psi_1(a)\|_{\mathbb{M}_m(OH)}\|\Psi_2(b)\|_{\mathbb{M}_n(OH)} \leq \|\Psi_1\|_{cb}\|\Psi_2\|_{cb}$$

and, taking the supremum on the left on the unit balls of M_1, M_2, we obtain the first part. The last part directly follows by definition. $\square$

8. THE SPLIT PROPERTY FOR INCLUSIONS OF W^*-ALGEBRAS AND CANONICAL NON-COMMUTATIVE EMBEDDINGS

In this section we provide the announced complete characterization of the split property for a factor–subfactor inclusion $N \subset M$ in terms of certain factorability property of the non-commutative L^2 embedding $\Phi_{2\lceil N} : N \to L^2(M)$ given in (11). The final part of this Section is devoted to extend this characterization to inclusions of W^*-algebras

which arise from Quantum Field Theory and have a-priori a non-trivial center, see e.g. [3, 27].

The following Lemma is a starting point.

Lemma 1. *Let M be a W^*-algebra and V an operator space. If $T \in \mathcal{F}_p(M, V)$ is a normal map then T has a decomposition*

$$T = A\mathcal{X}(b)$$

where b can be chosen in $S_p[M_*]$ and $\|A\|_{cb}\|b\|_{S_p[M^*]}$ is arbitrary close to $\varphi_p(T)$.

Proof. Suppose that $Tx = A\alpha f(x)\beta$ with $f \in \mathbb{M}_\infty(M^*)$ with $\|f\| \leq \varphi_p(T) + \varepsilon$. We can decompose f in its normal and singular parts f^n, f^s; then we have

$$Tx = A\alpha f^n(x)\beta + A\alpha f^s(x)\beta$$

where the summands on the l.h.s. are well defined, respectively normal and singular, operators in $\mathcal{F}_p(M, V)$ as $\|f^n\|, \|f^s\| \leq \|f\|$. We than have that $T - A\alpha f^n(\cdot)\beta$ is, at the same time, normal and singular and must be the zero mapping. Therefore

$$T = A\alpha f^n(x)\beta$$

and $\varphi_p(T) \leq \|f^n\| \leq \|f\| \leq \varphi_p(T) + \varepsilon$. $\qquad\square$

Now we are ready to prove the characterization of the split property for a factor–subfactor inclusion.
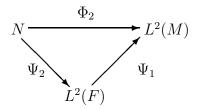
Theorem 5. *Let $N \subset M$ be an inclusion of W^*-factors with separable preduals and $\omega \in M_*$ a faithful state. Let $\Phi_i : M \to L^i(M^\circ)$, $i = 1, 2$ be the embeddings associated to a normal faithful state ω for M and given in (10), (11).*
The following statements are equivalent.
(i) $N \subset M$ is a split inclusion.
(ii) $\Phi_{1\lceil N} \in \mathcal{D}(N, (L^1(M^\circ))$.
(ii') The set $\{(\cdot\Omega, Ja\Omega) : a \in A, \|a\| < 1\}$ is Φ_1-decomposable (see [20], Definition 2.6 for this definition).
(iii) $\Phi_{2\lceil N} \in \mathcal{F}_2(N, (L^2(M))$.
(iii') The set $\{\Delta^{1/4}a\Omega : a \in A, \|a\| < 1\}$ is Φ_2-factorable.

Proof. Some of the above equivalences are immediate ([21]) or are contained in [20] so we prove the remaining ones.
$(i) \Rightarrow (iii)$ If there exists a type I interpolating factor F then Φ_2 factors according to

$$N \xrightarrow{\ \Phi_2\ } L^2(M)$$

with Ψ_2 from N to $L^2(F)$ and Ψ_1 from $L^2(F)$ to $L^2(M)$.

where Ψ_2 arises from $S_2[N_*]$ and Ψ_1 is bounded, see [4]. Moreover Ψ_1 is automatically completely bounded, see [33], Proposition 1.5.

$(iii) \Rightarrow (i)$ It is enough to show that, if $N \subset F$ with F a type I factor with separable predual, then Φ_1 extends to a completely positive map $\widetilde{\Phi}_1 : F \to L^1(M^\circ)$, see [4], Proposition 1.1. For the reader's convenience we split up this part of the proof in several steps.

Step 1. If $\Phi_{2\lceil N} \in \mathcal{F}_2(N, M)$ then $\Phi_{2\lceil N} = A\mathcal{X}(b)$ where b can be chosen in $S_2[N_*]$.

Proof. This directly follows by Lemma 1 as Φ_2 is a normal map.   □

Step 2. If $F \supset N$ is a Type I factor with separable predual then Φ_2 extends to a completely bounded normal map $\widetilde{\Phi} : F \to L^2(M^\circ)$.

Proof. As $\Phi_2(x) = A\alpha f(x)\beta$ for $x \in N$, where $f \in \mathbb{M}_\infty(N_*)$, we get a completely bounded normal map $\rho := \mathcal{X}(f) : N \to \mathbb{M}_\infty$. By Proposition 7, if $F \supset N$ is a Type I factor with separable predual we have a completely bounded normal extension $\tilde{\rho} : F \to \mathbb{M}_\infty$ to all of F. If we take $\tilde{f} := \mathcal{X}^{-1}(\tilde{\rho})$ we obtain the desired extension.   □

Unfortunately $\widetilde{\Phi}$ might be not positive, so we should recover a completely positive normal extension of $\Phi_{1\lceil N}$ to F via $\widetilde{\Phi}$.

Step 3. Let $\varphi : F \otimes M^\circ \to \mathbb{C}$ be the binormal bilinear form given by

$$\varphi(x \otimes y) := (\widetilde{\Phi}(x), \Delta^{1/4}y^*\Omega).$$

Then φ uniquely defines a bounded binormal form on all the C^*-algebra $F \otimes_{max} M^\circ$.

Proof. As $y \in M^\circ \to y^* \in M$ is a completely bounded antilinear map, we obtain the assertion by Proposition 8.   □

Step 4. φ can be decomposed in four positive binormal form.

Proof. We consider the universal envelopping von Neumann algebra $\mathcal{M}$ of $F \otimes_{max} M^{\circ}$ together with the central projection p relative to the binormal forms as described in [40], Theorem III.2.7. Then φ gets a normal form on the von Neumann algebra $\mathcal{M}_p$ and we can recover the desired decomposition via the Jordan decomposition described in [40], Theorem III.4.2. $\square$

Let φ_+ be the positive part of φ, then φ_+ gives rise to a cyclic F–M correspondence.

Step 5. φ_+, when restricted to $N \otimes M^{\circ}$, dominates ω given by

$$\omega(x \otimes y) := (x\Omega, Jy\Omega).$$

Proof. As $\mathrm{Re}\varphi = \varphi_+ - \varphi_-$ we get

$$\varphi_+(a) = \omega(a) + \varphi_-(a) \geq \omega(a)$$

whenever $a \in N \otimes M^{\circ}$. $\square$

Now we consider the GNS construction for the cyclic $F - M$ correspondence determined by φ_+ and obtain two normal commuting representations π, π° of F, M° respectively on a separable Hilbert space $\mathcal{H}$ and a vector $\xi \in \mathcal{H}$ (cyclic for $\pi(F) \vee \pi^{\circ}(M^{\circ})$) such that

$$\varphi_+(x \otimes y) = (\pi(x)\pi^{\circ}(y)\xi, \xi).$$

Then, as the restriction of φ_+ dominates ω, we have a positive element $T \in \pi(N)' \wedge \pi^{\circ}(M^{\circ})'$ such that, if $x \in N$, $y \in M^{\circ}$

$$\omega(x \otimes y) = (\pi(x)\pi^{\circ}(y)T\xi, T\xi).$$

Now we define $\widetilde{\Phi}_1 : F \to L^1(M^{\circ})$ given by

$$\widetilde{\Phi}_1(f) := (T\pi(f)T\pi^{\circ}(\cdot)\xi, \xi)$$

which is a completely positive normal map which extends Φ_1. This completes the proof. $\square$

As there exists examples of quasi-split inclusions which do not satisfy the split property ([12]), it is still unclear if the former can be characterized via the 2-factorable maps also for the case of inclusions of W^*-algebras with a non-trivial center. However in some interesting cases such as those arising from Quantum Field Theory we turn out to have the same characterization. We suppose that the net $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ of von Neumann algebras of local observables of a quantum theory acts on the Hilbert space $\mathcal{H}$ and satisfy all the usual assumptions (a priori without the split property) which are typical in Quantum Field Theory, $\Omega \in \mathcal{H}$ will be the vacuum vector which is cyclic for the net, see

e.g. [39]. We prefer to characterize the split property directly in terms of the shape of the sets $\Phi_i(\mathfrak{A}(\mathcal{O})_1)$, $i = 1, 2$ which are of particular interest in Quantum Field Theory, see [5, 7, 8, 9, 22].

Theorem 6. *Let* $\mathcal{O} \subset int(\widehat{\mathcal{O}})$ *be double cones in the physical space-time and* $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(\widehat{\mathcal{O}})$ *the corresponding inclusion of von Neumann algebras of observables.*
The following assertions are equivalent.
(i) $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(\widehat{\mathcal{O}})$ *is a split inclusion.*
(ii) The set $\{(\cdot a\Omega, \Omega) : a \in \mathfrak{A}(\mathcal{O})_1\} \subset (\mathfrak{A}(\widehat{\mathcal{O}})')_*$ *is* Φ_1-*decomposable.*
(iii) The set $\{\Delta^{1/4} a\Omega : a \in \mathfrak{A}(\mathcal{O})_1\} \subset \mathcal{H}$ *is* Φ_2-*factorable.*

Proof. If $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(\widehat{\mathcal{O}})$ is a split inclusion then (*ii*) and (*iii*) are true, see [4, 20, 21]. Conversely, if (*ii*) or (*iii*) are satisfied, then Φ_1 is extendible, see [20] for the implication (*ii*) $\Rightarrow$ (*i*) or the proof of the last Theorem for the implication (*iii*) $\Rightarrow$ (*i*). Hence the map

$$\eta : a \otimes b \in \mathfrak{A}(\mathcal{O}) \otimes \mathfrak{A}(\widehat{\mathcal{O}})' \to ab \in \mathfrak{A}(\mathcal{O}) \vee \mathfrak{A}(\widehat{\mathcal{O}})'$$

extends to a normal homomorphism of $\mathfrak{A}(\mathcal{O})\overline{\otimes}\mathfrak{A}(\widehat{\mathcal{O}})$ onto $\mathfrak{A}(\mathcal{O})\vee\mathfrak{A}(\widehat{\mathcal{O}})$. But this homomorphism is in fact an isomorphism by an argument exposed in [3], pagg. 129-130. Moreover, as $\mathfrak{A}(\mathcal{O}) \wedge \mathfrak{A}(\widehat{\mathcal{O}})$ is properly infinite ([25]), the assertion now follows by Corollary 1 of [12]. $\square$

For the applications to Quantum Field Theory, the split property turns out to be completely characterized by properties regarding the non-commutative embeddings $\Phi_{i,\omega}$, $i = 1, 2$ given in (10), (11) which do not depend on the choice of the faithful state $\omega \in M$. We wish to note that the characterizations listed above could be of interest in particular for the applications to theories living on a curved space-time where there is no privileged state as the vacuum, see e.g. [41].

9. REMARKS ABOUT THE CANONICAL EMBEDDINGS $M \hookrightarrow L^p(M)$ WITH p ARBITRARY

In this paper we have studied properties relative to symmetric embedding $\Phi_2 : M \to L^2(M)$. Moreover a complete characterization of the split property has been given for an inclusion $N \subset M$. If one collect the above results together with those contained in [4, 20], we have a situation relative to the embeddings Φ_i, $i = 1, 2$ which seems to be satisfactory enough. On the other hand, it would be of interest to consider other canonical embeddings which can be constructed for arbitrary p. This can be made in the following way. We leave fixed a faithful state $\omega \in M_*$ and consider the crossed product $R := M \times_{\sigma^\omega} \mathbb{R}$ where σ^ω is the modular group relative to ω. R has a faithful semifinite normal

trace τ and the dual action θ (scaling automorphisms) of $\mathbb{R}$ satisfies $\tau \circ \theta_s = e^{-s}\tau$. If $1 \le p < +\infty$, one can identify the L^p spaces with the spaces of all τ-measurable operators h affiliated to R which satisfy, for $s \in \mathbb{R}$

$$\theta_s(h) = e^{-\frac{s}{p}}h.$$

If we fix a vector $h_\omega \in L^2$ which represents ω, the canonical symmetric non-commutative embeddings $\Phi_p : M \to L^p(M)$, $1 \le p < +\infty$ assume the following simple form

$$\Phi_p(x) := h_\omega^{\frac{1}{2p}} x h_\omega^{\frac{1}{2p}}. \tag{14}$$

It easy to show that the last definition coincide with our previous one in the case when $p = 1, 2$. As a first step, we have shown that, if $p = 1, 2$, the above embeddings are completely bounded when a suitable operator space structure is kept fixed on $L^p(M)$. These are precisely

(a) the canonical (pre)dual structure of $L^1(M)$ when it is consider-ated as the predual of M°, the opposite algebra of M, for the embedding Φ_1,
(b) the Pisier OH structure on $L^2(M)$ for the embedding Φ_2.

Moreover, properties of restrictions $\Phi_{1\lceil N}$, $\Phi_{2\lceil N}$, characterize the split property for the inclusion $N \subset M$. Also in this case the operator space structures considered on $L^1(M)$, $L^2(M)$ play a crucial role. On all of $L^p(M)$, it will be a lot of interpolating operator space structures. Unfortunately this picture is not yet understood in the full generality, see [34]. In the light of the above considerations, it would be of interest to extend our characterization of the split property in terms of the other embeddings $\Phi_p : M \hookrightarrow L^p(M)$ (14). We remark that also non-symmetric embeddings could be considered as well, see [26], Section 7. We hope to return on these interesting questions somewhere else.

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